

# MOTIVIC DECOMPOSITIONS OF TWISTED FLAG VARIETIES AND REPRESENTATIONS OF HECKE-TYPE ALGEBRAS

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**ABSTRACT.** Let  $G$  be a split semisimple linear algebraic group over a field  $k_0$ . Let  $E$  be a  $G$ -torsor over a field extension  $k$  of  $k_0$ . Let  $\mathbf{h}$  be an algebraic oriented cohomology theory in the sense of Levine-Morel. Consider a twisted form  $E/B$  of the variety of Borel subgroups  $G/B$  over  $k$ .

Following the Kostant-Kumar results on equivariant cohomology of flag varieties we establish an isomorphism between the Grothendieck groups of the  $\mathbf{h}$ -motivic subcategory generated by  $E/B$  and the category of finitely generated projective modules of certain Hecke-type algebra  $H$  which depends on the root datum of  $G$ , on the torsor  $E$  and on the formal group law of the theory  $\mathbf{h}$ .

In particular, taking  $\mathbf{h}$  to be the Chow groups with finite coefficients  $\mathbb{F}_p$  and  $E$  to be a generic  $G$ -torsor we prove that all indecomposable submodules of an affine nil-Hecke algebra  $H$  of  $G$  with coefficients in  $\mathbb{F}_p$  are isomorphic to each other and correspond to the (non-graded) generalized Rost-Voevodsky motive for  $(G, p)$ .

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## 1. INTRODUCTION

Let  $G$  be a split semisimple linear algebraic group over a field  $k_0$  and let  $E$  be a  $G$ -torsor over a field extension  $k$  of  $k_0$ . Consider a twisted form  $E/B$  of the variety of Borel subgroups  $G/B$  of  $G$  over  $k$ . Observe that  $E/B$  is a smooth projective variety over  $k$  that in general has no rational points. For example, for  $G = PGL_p$  and a non-split  $E$ ,  $E/B$  is a variety of complete flags of ideals in a central simple division algebra of a prime degree  $p$  over  $k$ .

Following [14, §64] consider the category of graded Chow motives  $CM(k, \mathbb{F}_p)$  of smooth projective varieties over  $k$  with finite coefficients  $\mathbb{F}_p$ . According to [25, Theorem 5.17] the motive  $[E/B]$  of  $E/B$  splits as a direct sum of Tate twists of some indecomposable motive  $\mathcal{R}$ , a generalization of the Rost-Voevodsky motive, i.e.,

$$[E/B] \simeq \bigoplus_{i \in I} \mathcal{R}(i).$$

Hence, if  $\langle [E/B] \rangle$  denotes a pseudo-abelian subcategory generated by the motive  $[E/B]$ , i.e., a minimal pseudo-abelian category containing  $[E/B]$ , then

$$\langle [E/B] \rangle = \langle \mathcal{R}(i) \rangle_{i \in I}.$$

Observe that in the non-graded case (in the category of motives  $CM_*(k, \mathbb{F}_p)$  of [14, §64]) all Tate twists become isomorphic and we have  $\langle [E/B]_* \rangle = \langle \mathcal{R}_* \rangle$ , where  $[E/B]_*$  and  $\mathcal{R}_*$  denote the respective non-graded motives.

The motive  $\mathcal{R}$  has several remarkable properties (see [25, §5]). If  $p$  is not a torsion prime of  $G$ , then  $\mathcal{R}$  coincides with the motive of a point, so  $\langle [E/B] \rangle$  is generated by Tate twists  $\mathbb{F}_p(i)$ ,  $i = 0.. \dim G/B$ . While being indecomposable over  $k$ , the motive  $\mathcal{R}$  becomes isomorphic to a direct sum of Tate twists over a splitting field  $\bar{k}$  of  $E$  (as  $\bar{k}$  one can always take an algebraic closure of  $k$  or a function field of  $E/B$ ). Moreover, the Poincaré polynomial of  $\mathcal{R}$  over  $\bar{k}$  is given by an explicit polynomial. For example, if  $G$  is an exceptional group of type  $F_4$  and  $p = 3$ , then  $\mathcal{R}|_{\bar{k}} \simeq \mathbb{F}_3 \oplus \mathbb{F}_3(4) \oplus \mathbb{F}_3(8)$  for a non-split  $E$ .

Only very few facts are known concerning the subcategory  $\langle [E/B] \rangle$  of Chow motives with integer coefficients. An integer version of the motive  $\mathcal{R}$  was introduced and discussed in [31]; in [5], [10], [27] it was shown that  $\langle [E/B] \rangle$  is not Krull-Schmidt (the uniqueness of a direct sum decomposition fails).

In the present paper we consider the category of  $\mathbf{h}$ -motives with coefficients in a commutative ring  $\mathbf{R} = \mathbf{h}(k)$ , where  $\mathbf{h}$  is any algebraic oriented cohomology theory over  $k$  in the sense of Levine-Morel [24], e.g., Chow ring with integer or finite coefficients,  $K$ -theory, algebraic cobordism  $\Omega$  with coefficients in the Lazard ring. Let  $\langle [E/B] \rangle_{\mathbf{h}}$  (resp.  $\langle [E/B]_* \rangle_{\mathbf{h}}$ ) denote its pseudo-abelian subcategory generated by the (resp. non-graded)  $\mathbf{h}$ -motive of  $E/B$ . Our main result (Theorem 8.1) establishes isomorphisms between the Grothendieck groups

$$(1) \quad K_0(\langle [E/B] \rangle_{\mathbf{h}}) \simeq K_0(\overline{\mathbf{D}}_F^{(0)}) \quad \text{and} \quad K_0(\langle [E/B]_* \rangle_{\mathbf{h}}) \simeq K_0(\overline{\mathbf{D}}_F)$$

of the category  $\langle [E/B] \rangle_{\mathbf{h}}$  (resp.  $\langle [E/B]_* \rangle_{\mathbf{h}}$ ) and the category of finitely generated projective modules over a certain  $\mathbf{R}$ -algebra  $\overline{\mathbf{D}}_F^{(0)}$  (resp.  $\overline{\mathbf{D}}_F$ ). More precisely, the algebra  $\overline{\mathbf{D}}_F^{(0)}$  is the degree 0 component of the  $\mathbf{R}$ -algebra  $\overline{\mathbf{D}}_F$  defined using the formal push-pull operators (see Definition 7.5); it depends on the root datum of  $G$ , on the formal group law  $F$  of the theory  $\mathbf{h}$  and on the subring of rational cycles in  $\mathbf{h}(G/B)$ .

If  $E$  is a generic  $G$ -torsor, then  $\overline{\mathbf{D}}_F$  can be replaced by the formal affine Demazure algebra  $\mathbf{D}_F$ . The theory of such algebras and formal push-pull operators has been recently developed in [6], [19], [7], [8], [9] motivated by Bernstein-Gelfand-Gelfand [1], Demazure [11], [12], Bressler-Evens [2], [3], Kostant-Kumar [22], [21], Brion [4], Totaro [29] and Edidin-Graham [13]. The key properties of  $\mathbf{D}_F$  are

- It is a free module over the  $T$ -equivariant oriented cohomology ring  $\mathbf{S} = \mathbf{h}_T(k)$  of a point, where  $T$  is a split maximal torus in  $G$  [7].
- Its  $\mathbf{S}$ -dual  $\mathbf{D}_F^* = \text{Hom}_{\mathbf{S}}(\mathbf{D}_F, \mathbf{S})$  is isomorphic to the  $T$ -equivariant oriented cohomology ring  $\mathbf{h}_T(G/B)$  of  $G/B$  [9].
- Its structure (generators and relations) is very close to those of the affine Hecke algebra [19].

For example, if  $\mathbf{h}(-) = CH(-; \mathbb{F}_p)$  is the Chow ring with finite coefficients, then  $\mathbf{D}_F^* \simeq CH_T(G/B; \mathbb{F}_p)$  is the  $T$ -equivariant Chow ring and  $\mathbf{D}_F = \mathbf{H}_{nil,p}$  is the affine nil-Hecke algebra over  $\mathbb{F}_p$  (in the notation of Ginzburg [17, §12]) which is a free module of rank  $|W|$  over the polynomial ring  $\mathbf{S} = \mathbb{F}_p[x_1, \dots, x_n]$ , where  $n$  is the rank of  $G$  and  $W$  is the Weyl group.

For generic  $E$  the isomorphisms (1) then turn into (see Corollary 8.4)

$$K_0(\langle \mathcal{R}(i) \rangle_{i \in I}) \simeq K_0(\mathbf{H}_{nil,p}^{(0)}) \quad \text{and} \quad K_0(\langle \mathcal{R}_* \rangle) \simeq K_0(\mathbf{H}_{nil,p}),$$

where the Tate twists  $\mathcal{R}(i)$  correspond to indecomposable  $\mathbf{H}_{nil,p}^{(0)}$ -submodules. Moreover, there is a ring isomorphism

$$\mathbf{H}_{nil,p} \simeq \text{Mat}_{|W|/r}(\text{End}(P_*)),$$

where  $P_*$  is the projective  $\mathbf{H}_{nil,p}$ -module corresponding to  $\mathcal{R}_*$  and  $r$  is the  $p$ -part of the product of  $p$ -exceptional degrees of the group  $G$ .

The latter isomorphism specialized to  $G = SL_n$  and  $\mathbf{h} = CH$  gives [26, 3.1.16] and [23, Prop. 3.5]. Indeed, in this case  $E$  is split,  $r = 1$  and  $\mathbf{S}$  is a free  $\mathbf{S}^W$ -module with  $\mathbf{h}(G/B) \simeq \mathbf{R} \otimes_{\mathbf{S}^W} \mathbf{S}$ . Then by Lemma 7.3 one obtains that

$$\mathbf{H}_{nil,p} \simeq \mathbf{S}^W \otimes_{\mathbf{R}} \text{Mat}_{n!}(\mathbf{R}) \simeq \text{Mat}_{n!}(\mathbf{S}^W).$$

In the paper we restrict ourselves to varieties  $E/B$  of Borel subgroups only. However, by [5]  $B$  can be replaced by any special parabolic subgroup  $P$  without affecting the isomorphism (1) for non-graded motives. For instance, for  $G = PGL_n$ ,  $\mathbf{h} = CH(-; \mathbb{Z})$  and  $E$  corresponding to a generic central division algebra  $A$  of degree  $n$  we get

$$K_0(\langle [SB(A)]_* \rangle) \simeq K_0(\mathbf{H}_{nil,\mathbb{Z}}),$$

where  $SB(A)$  is the Severi-Brauer variety of  $A$  and  $\mathbf{H}_{nil,\mathbb{Z}}$  is the affine nil-Hecke algebra for  $PGL_n$  with integer coefficients.

The paper is organized as follows. In section 2 we recall definitions and basic facts concerning Borel-Moore homology  $\mathbf{h}$  and the respective category of  $\mathbf{h}$ -motives. We state a version of the Künneth isomorphism for cellular spaces. In the next section we generalize it to the equivariant setting. In section 4 we introduce the convolution product on the equivariant cohomology of products and study its properties. In the next section we identify the equivariant cohomology of  $G$  with respect to the convolution product with the endomorphism ring of  $T$ -equivariant cohomology of  $G/B$  and then in section 6 with the formal affine Demazure algebra. In section 7 we introduce the notion of a rational algebra of push-pull operators  $\overline{\mathbf{D}}_F$

and identify it with the subring of rational endomorphisms. In the last section we prove isomorphisms (1) and provide applications and examples.

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## 2. ORIENTED (CO-)HOMOLOGY

We recall definitions of an algebraic oriented Borel-Moore homology and of the respective category of correspondences. We also recall a version of the Künneth isomorphism for cellular spaces (Lemmas 2.4 and 2.5).

Fix a smooth scheme  $S$  over a field  $k$ . Let  $Sch_S$  denote the category of finite type quasi-projective separated  $S$ -schemes and let  $Sm_S$  denote its full subcategory consisting of smooth quasi-projective  $S$ -schemes.

Following [24, Def. 5.1.3] consider an oriented graded Borel-Moore homology theory  $\mathbf{h}_\bullet$  defined on some admissible [24, (1.1)] subcategory  $\mathcal{V}$  of  $Sch_S$ . So that there are pull-backs  $f^*: \mathbf{h}_\bullet(X) \rightarrow \mathbf{h}_{\bullet+d}(Y)$  for l.c.i. morphisms  $f: Y \rightarrow X$  in  $\mathcal{V}$  of relative dimension  $d$  and push-forwards  $f_*: \mathbf{h}_\bullet(Y) \rightarrow \mathbf{h}_\bullet(X)$  for projective morphisms  $f: X \rightarrow Y$  in  $\mathcal{V}$ . According to [24, Prop. 5.2.1] the Borel-Moore homology  $\mathbf{h}_\bullet$  restricted to  $Sm_S$  defines an algebraic oriented cohomology theory  $\mathbf{h}^\bullet$  (with values in the category of graded commutative rings with unit) in the sense of [24, Def. 1.1.2] by

$$\mathbf{h}^{\dim_S X - \bullet}(X) := \mathbf{h}_\bullet(X), \quad X \in Sm_S.$$

If the (co-)dimension is clear from the context we will write simply  $\mathbf{h}(X)$ .

Following [14, §63] and [31, §2] we define the category of  $\mathbf{h}$ -correspondences  $\mathbf{h}\text{-}CR(S)$  over  $S$ . The objects are pairs  $([X \rightarrow S], i)$ , where  $[X \rightarrow S]$  is an isomorphism class of a smooth projective map  $X \rightarrow S$  and  $i \in \mathbb{Z}$ . The morphisms are defined by

$$Hom_{\mathbf{h}\text{-}CR(S)}([Y \rightarrow S], i), ([X \rightarrow S], j) := \bigoplus_l Hom_{i-j}([Y_l \rightarrow S], [X \rightarrow S]),$$

taken over all connected components  $Y_l$  of  $Y$ , where

$$Hom_\bullet([Y_l \rightarrow S], [X \rightarrow S]) := \mathbf{h}_{\dim_S Y_l + \bullet}(Y_l \times_S X).$$

The composition of morphisms is given by the correspondence product. Namely, if  $p_i: X_1 \times_S X_2 \times_S X_3 \rightarrow X_j \times_S X_{j'}$  denotes the projection obtained by removing the  $i$ -th coordinate, then given  $\alpha \in \mathbf{h}(X_1 \times_S X_2)$  and  $\beta \in \mathbf{h}(X_2 \times_S X_3)$  we set

$$(2) \quad \beta \circ \alpha := (p_2)_*(p_1^*(\beta) \cdot p_3^*(\alpha)) \in \mathbf{h}(X_1 \times_S X_3).$$

The idempotent completion of  $\mathbf{h}\text{-}CR(S)$  denoted by  $\mathbf{h}\text{-}M(S)$  is called the category of  $\mathbf{h}$ -motives. We simply write  $[X]$  for the respective class in  $\mathbf{h}\text{-}M(S)$ .

We also consider the non-graded version of  $\mathbf{h}\text{-}CR(S)$  and of  $\mathbf{h}\text{-}M(S)$  denoted by  $\mathbf{h}\text{-}CR_*(S)$  and  $\mathbf{h}\text{-}M_*(S)$  respectively, where the objects are given by isomorphism classes  $[X \rightarrow S]$  of smooth projective maps and the morphisms are defined by

$$Hom_{\mathbf{h}\text{-}CR_*(S)}([Y \rightarrow S], [X \rightarrow S]) := \mathbf{h}(Y \times_S X).$$

**Definition 2.1.** (cf. [24, (CD')]) Let  $X$  be smooth projective over  $S$ . Suppose that there is a filtration by proper closed subschemes

$$\emptyset = X_{-1} \subset X_0 \subset X_1 \subset \dots \subset X_n = X$$

such that

- each irreducible component  $X_{ij}$  of  $X_i \setminus X_{i-1}$  is a locally trivial affine fibration over  $S$  of rank  $d_{ij}$ , and
- the closure of  $X_{ij}$  in  $X$  admits a resolution of singularities  $\tilde{X}_{ij} \rightarrow \overline{X}_{ij}$  over  $S$ ; we set  $g_{ij}: \tilde{X}_{ij} \rightarrow \overline{X}_{ij} \hookrightarrow X$  and, therefore,  $(g_{ij})_*(1_{\tilde{X}_{ij}}) \in \mathbf{h}_{d_{ij}}(X)$ .

We call such  $X$  (together with the filtration) a cellular space over  $S$ .

**Definition 2.2.** We say that the theory  $\mathbf{h}$  satisfies the cellular decomposition (CD) property if given a cellular space  $X$  over  $S$  the respective elements  $(g_{ij})_*(1_{\tilde{X}_{ij}})$  form a  $\mathbf{h}(S)$ -basis of  $\mathbf{h}(X)$ .

**Example 2.3.** The property (CD) holds for any oriented Borel-Moore homology  $\mathbf{h}$  over a field  $k$  of characteristic 0.

Indeed, the same reasoning as in [14, Thm. 66.2] shows that for every  $Z \in Sm_S$  there is an isomorphism

$$\sum (g_{ij})_*(1) \times id_Z: \bigoplus_{ij} CH_{\bullet-d_{ij}}(Z) \rightarrow CH_{\bullet}(Z \times_S X).$$

By the Yoneda lemma (cf. [14, Lemma 63.9]) the latter induces an isomorphism in the category  $CM(S)$  (cf. [14, Cor. 66.4]).

Following [30, §2] consider the specialization functor  $\Omega\text{-}M(S) \rightarrow CM(S)$ ,  $[f: Y \rightarrow X] \mapsto f_*(1_Y)$ . It is surjective on the classes of objects and morphisms. Moreover, for every  $X$  the kernel of

$$\Omega_{\dim_S X}(X \times_S X) \longrightarrow CH_{\dim_S X}(X \times_S X)$$

is  $\Omega_{\geq 1}(k) \cdot \Omega_{\bullet}(X \times_S X)$  by [24, Rem. 4.5.6]. Hence for every  $y$  in this kernel

$$y^{\circ(\dim_S X + 1)} \in \Omega_{\dim_S X}(X \times_S X) \cap (\Omega_{\geq (\dim_S X + 1)}(k) \cdot \Omega_{\bullet}(X \times_S X)).$$

So  $y = 0$  since  $\Omega_{< 0}(Y) = 0$ . Therefore, the kernel of

$$\text{End}_{\Omega\text{-}M(S)}([X], i) \rightarrow \text{End}_{CM(S)}([X], i)$$

consists of nilpotents.

Finally, by [30, Lemma 2.1] the isomorphism  $\sum_{ij} (g_{ij})_*(1)$  in  $CM(S)$  can be lifted to an isomorphism in the category  $\Omega\text{-}M(S)$ . Specializing it via  $\Omega \rightarrow \mathbf{h}$  we obtain the desired isomorphism.

**Lemma 2.4.** Assume that  $\mathbf{h}$  satisfies the property (CD). Let  $X$  be a cellular space over  $S$ . Then there is an isomorphism in  $\mathbf{h}\text{-}M(S)$

$$\sum_{ij} (g_{ij})_*(1_{\tilde{X}_{ij}}): \bigoplus_{ij} ([S], d_{ij}) \rightarrow [X],$$

where  $(g_{ij})_*(1_{\tilde{X}_{ij}}) \in \mathbf{h}_{d_{ij}}(X) = \text{Hom}_{\mathbf{h}\text{-}M(S)}([S], d_{ij}), [X])$ .

*Proof.* Transversal base change implies that there is an isomorphism

$$\sum (g_{ij})_*(1) \times id_Z: \bigoplus_{ij} \mathbf{h}_{\bullet-d_{ij}}(Z \times_S S) \rightarrow \mathbf{h}_{\bullet}(Z \times_S X)$$

for any  $Z$  smooth projective over  $S$ . So by the Yoneda lemma (cf. [14, Lemma 63.9]) it induces an isomorphism in  $\mathbf{h}\text{-}M(S)$  (cf. [14, Cor. 66.4]).  $\square$

**Lemma 2.5.** *Assume that  $\mathbf{h}$  satisfies the property (CD). Let  $X$  be a cellular space over  $S$ . The pairing  $(\cdot, \cdot): \mathbf{h}(X) \otimes_{\mathbf{h}(S)} \mathbf{h}(X) \rightarrow \mathbf{h}(S)$  given by  $(a, b) = p_*(ab)$  is non-degenerate and the map*

$$f: (\mathbf{h}(X \times_S X), \circ) \rightarrow \text{End}_{\mathbf{h}(S)} \mathbf{h}(X) \quad \text{given by } a \mapsto f_a, f_a(x) = (p_2)_*(p_1^*(x) \cdot a)$$

*is an  $\mathbf{h}(S)$ -linear isomorphism of graded rings. In particular, it gives an  $\mathbf{h}(S)$ -linear isomorphism*

$$(\mathbf{h}_{\dim_S X}(X \times_S X), \circ) \simeq \text{End}_{\mathbf{h}-M(S)}(X).$$

Observe that the endomorphism ring of  $\mathbf{h}(S)$ -linear operators  $\text{End}_{\mathbf{h}(S)}(\mathbf{h}(X))$  is a graded ring. Its  $n$ -th graded component consists of operators increasing the codimension by  $n$ . By definition the subring of degree-0 operators (preserving the codimension) coincides with  $\text{End}_{\mathbf{h}-M(S)}(X)$ .

*Proof.* By the previous lemma there is an isomorphism

$$\bigoplus_{ij} \mathbf{h}(S) = \bigoplus_{k=-\infty}^{\infty} \text{Hom}([S], k, \oplus_{ij}([S], d_{ij})) \xrightarrow{\sim} \bigoplus_{k=-\infty}^{\infty} \text{Hom}([S], k, [X]) = \mathbf{h}(X),$$

where each component is given by  $x \mapsto x \cdot (g_{ij})_*(1)$ . Let  $\sum_{ij} a_{ij}: [X] \rightarrow \oplus_{ij}([S], d_{ij})$  be the inverse isomorphism in  $\mathbf{h}-M(S)$ . Observe that

$$a_{ij} \in \text{Hom}([X], ([S], d_{ij})) = \mathbf{h}_{\dim(X/S)-d_{ij}}(X).$$

Since  $a_{ij} \circ (g_{ij})_*(1) = p_*(a_{ij} \cdot (g_{ij})_*(1)) = \delta_{i,j}$ , the pairing  $(\cdot, \cdot)$  is non-degenerate.

The pairing  $(\cdot, \cdot)$  gives an isomorphism  $\mathbf{h}(X) \rightarrow \text{Hom}_{\mathbf{h}(S)}(\mathbf{h}(X), \mathbf{h}(S))$  and, hence, an isomorphism  $\text{End}_{\mathbf{h}(S)} \mathbf{h}(X) \xrightarrow{\sim} \mathbf{h}(X) \otimes_{\mathbf{h}(S)} \mathbf{h}(X)$ . Consider the composition

$$\rho: \mathbf{h}(X \times_S X) \xrightarrow{f} \text{End}_{\mathbf{h}(S)} \mathbf{h}(X) \xrightarrow{\sim} \mathbf{h}(X) \otimes_{\mathbf{h}(S)} \mathbf{h}(X)$$

and a map  $\pi: \mathbf{h}(X) \otimes_{\mathbf{h}(S)} \mathbf{h}(X) \rightarrow \mathbf{h}(X \times_S X)$  given by  $\pi(a \otimes b) = p_1^*(a) \cdot p_2^*(b)$ .

By definition, we have

$$f_{p_1^*(a)p_2^*(b)}(x) = (p_2)_*(p_1^*(x)p_1^*(a)p_2^*(b)) = (x, a)b.$$

Hence,  $\rho(\pi(a \otimes b)) = a \otimes b$  and the map  $\rho$  is surjective. By the property (CD) for  $X \times_S X \rightarrow X$ ,  $\mathbf{h}(X \times_S X)$  is a free  $\mathbf{h}(X)$ -module of rank  $rk_{\mathbf{h}(S)} \mathbf{h}(X)$ . Thus,  $\rho$  is a surjective homomorphism between free modules of the same rank, hence, it is an isomorphism.  $\square$

Let  $C$  be any pseudo-abelian category. For an object  $X \in C$  consider a subcategory  $\langle X \rangle$  generated by  $X$ , i.e., the smallest pseudo-abelian subcategory of  $C$  that contains  $X$ .

**Lemma 2.6.** *The category  $\langle X \rangle$  is equivalent to the category of finitely generated projective  $\text{End}_C(X)$ -modules.*

*Proof.* Denote  $\text{End}_C(X)$  by  $R$ . Every element  $Y$  of  $\langle X \rangle$  is isomorphic to the image  $p(X^{\oplus n})$  of some idempotent  $p \in \text{End}_C(X^{\oplus n}) = \text{Mat}_n(R)$  and  $\text{Hom}_C(X, p(X^{\oplus n})) = p(R^n)$ . Note that

$$\text{Hom}_C(p(X^{\oplus n}), p'(X^{\oplus n'})) = p' \text{Hom}_C(X^{\oplus n}, X^{\oplus n'}) p = \text{Hom}_R(p(R^n), (p' R^{n'})).$$

Then the functor  $Y \mapsto \text{Hom}_C(X, Y)$  establishes an equivalence between  $\langle X \rangle$  and the category of finitely generated projective right  $R$ -modules.  $\square$

**Corollary 2.7.** *The category  $\langle [E/B] \rangle_{\mathbf{h}}$  (resp.  $\langle [E/B]_* \rangle_{\mathbf{h}}$ ) is equivalent to the category of finitely generated projective modules over the endomorphism ring of the (resp. non-graded)  $\mathbf{h}$ -motive of  $E/B$ .*

### 3. THE EQUIVARIANT KÜNNETH ISOMORPHISM

In the present section we introduce an equivariant Borel-Moore homology following [7, §2] and [18]. We provide an equivariant analogue of the Künneth isomorphism (Lemma 3.7).

Let  $G$  be a smooth group scheme over  $S$ . Consider an admissible subcategory  $\mathcal{V}^G$  of the category of  $G$ -varieties  $X \in \text{Sch}_S$  with  $G$ -equivariant morphisms. By a  $G$ -equivariant oriented (graded) Borel-Moore homology theory we will call an additive functor  $\mathbf{h}_\bullet^G$  from  $\mathcal{V}^G$  to graded abelian groups such that

1. There are pull-backs for l.c.i. maps and push-forwards for projective maps that satisfy

(TS) (l.c.i. base change) For a Cartesian square  $\begin{array}{ccc} X' & \xrightarrow{f'} & Y' \\ g' \downarrow & & \downarrow g \\ X & \xrightarrow{f} & Y \end{array}$  where  $f$  (hence  $f'$ ) is l.c.i. and  $g$  (hence  $g'$ ) is projective, we have  $f^*g_* = g'_*(f')^*$ .

- (Loc) (localization) If  $U \subset X$  is an open  $G$ -equivariant embedding with  $Z = X \setminus U$ , then there is a right exact sequence:

$$\mathbf{h}_\bullet^G(Z) \rightarrow \mathbf{h}_\bullet^G(X) \rightarrow \mathbf{h}_\bullet^G(U) \rightarrow 0.$$

2. The functor  $\mathbf{h}_\bullet^G$  restricted to  $\text{Sm}_S$  defines a graded  $G$ -equivariant oriented cohomology theory  $\mathbf{h}_G^\bullet$  in the sense of [9] (we refer to [9, §2, A1-9] for the precise definition) by

$$\mathbf{h}_G^{\dim_S X - \bullet}(X) := \mathbf{h}_\bullet^G(X), \quad X \in \text{Sm}_S.$$

In addition to the axioms of [9, §2] we require that  $\mathbf{h}_G^\bullet$  satisfies the following stronger version of the homotopy invariance axiom:

- (HI) (extended homotopy invariance) Let  $p: Y \rightarrow X$  be a  $G$ -equivariant torsor of a vector bundle of rank  $r$  over  $X$ , then the pull-back induced by projection

$$p^*: \mathbf{h}_G^\bullet(X) \rightarrow \mathbf{h}_G^\bullet(Y)$$

is an isomorphism.

If a variety is smooth we will always use the cohomology notation.

**Example 3.1.** Given a linear algebraic group  $G$  over a field  $k$  of characteristic zero an example of such  $G$ -equivariant Borel-Moore homology theory  $\mathbf{h}_\bullet^G$  was constructed in [18] as follows.

Consider a system of  $G$ -representations  $V_i$  and its open subsets  $U_i \subseteq V_i$  such that

- $G$  acts freely on  $U_i$  and the quotient  $U_i/G$  exists as a scheme over  $k$ ,
- $V_{i+1} = V_i \oplus W_i$  for some representation  $W_i$ ,
- $U_i \subseteq U_i \oplus W_i \subseteq U_{i+1}$ , and  $U_i \oplus W_i \rightarrow U_{i+1}$  is an open inclusion, and
- $\text{codim}(V_i \setminus U_i)$  strictly increases.

Such a system is called a good system of representations of  $G$ .

Let  $X \in Sch_k$  be a  $G$ -variety. Following [18, §3 and §5] the inverse limit induced by pull-backs

$$\varprojlim_i \mathbf{h}_{\bullet - \dim G + \dim U_i}(X \times^G U_i), \quad X \times^G U_i = (X \times_k U_i)/G,$$

does not depend on the choice of the system  $(V_i, U_i)$  and, hence, defines the  $G$ -equivariant oriented homology group  $\mathbf{h}_{\bullet}^G(X)$ .

In the present paper we will extensively use the following property (cf. [9, §2, A6]) of an equivariant theory

(Tor) Let  $X \rightarrow X/G$  be a  $G$ -torsor over  $S$  and a  $G'$ -equivariant map for some group scheme  $G'$  over  $S$ . Then there is an isomorphism

$$\mathbf{h}_{G \times G'}^{\bullet}(X) \xrightarrow{\sim} \mathbf{h}_{G'}^{\bullet}(X/G).$$

that is natural with respect to the maps of pairs

$$(\phi, \gamma): (X, G \times G') \rightarrow (X_1, G_1 \times G'_1), \quad \phi(x \cdot (g, g')) = \phi(x) \cdot \gamma(g, g').$$

Observe that the theory of Example 3.1 satisfies this property by [18, Prop. 27].

We have the following equivariant analogues of Definitions 2.1 and 2.2

**Definition 3.2.** Let  $X \in \mathcal{V}^G$ . Suppose that there is a filtration by  $G$ -equivariant proper closed subschemes

$$\emptyset = X_{-1} \subset X_0 \subset X_1 \subset \dots \subset X_n = X$$

such that

- each irreducible component  $X_{ij}$  of  $X_i \setminus X_{i-1}$  is a  $G$ -equivariant (locally trivial) affine fibration over  $S$  of rank  $d_{ij}$ , and
- the closure of  $X_{ij}$  in  $X$  admits a  $G$ -equivariant resolution of singularities  $g_{ij}: \tilde{X}_{ij} \rightarrow \overline{X}_{ij}$  over  $S$ .

We call such  $X$  (together with the filtration) a  $G$ -equivariant cellular space over  $S$ .

**Definition 3.3.** We say that the equivariant theory  $\mathbf{h}^G$  satisfies the cellular decomposition (CD) property if given a  $G$ -equivariant cellular space  $X$  over  $S$  the respective elements  $(g_{ij})_*(1_{\tilde{X}_{ij}})$  form a  $\mathbf{h}^G(S)$ -basis of  $\mathbf{h}^G(X)$ .

**Lemma 3.4.** Suppose a morphism  $f: X \rightarrow Y$  in  $Sm_k$  factors as  $f: X \xrightarrow{z} L \xrightarrow{j} Y$  where  $p: L \rightarrow X$  is a vector bundle,  $z: X \rightarrow L$  is a zero section and  $j$  is an open embedding.

Then for every projective map  $a: Y' \rightarrow Y$  and  $X' = X \times_Y Y'$  the following diagram of pull-back and push-forward maps commutes (we omit the grading)

$$\begin{array}{ccc} \mathbf{h}(X') & \xrightarrow{a'_*} & \mathbf{h}(X) \\ f'^* \uparrow & & \uparrow f^* \\ \mathbf{h}(Y') & \xrightarrow{a_*} & \mathbf{h}(Y) \end{array}$$

*Proof.* Observe that the map  $f': X' \rightarrow Y'$  factors as  $X' \xrightarrow{z'} L \times_Y Y' \xrightarrow{j'} Y'$  where  $z'$  is the zero section of the vector bundle  $p': L' = L \times_Y Y' \rightarrow X'$  and  $j'$  is an open embedding. Let  $b$  denote the canonical map  $L' \rightarrow L$ . Since  $j$  and  $j'$  are flat,



we have  $j^*a_* = b_*j'^*$  by the l.c.i. base change for oriented theories. Note that by the homotopy invariance  $z^* = (p^*)^{-1}$  and  $z'^* = (p'^*)^{-1}$ . Since  $p$  and  $p'$  are flat,  $p^*a'_* = b_*p'^*$ . Then  $z^*b_* = a'_*z'^*$  and

$$f^*a_* = z^*j^*a_* = z^*b_*j'^* = a'_*z'^*j'^* = a'_*f'^*. \quad \square$$

**Remark 3.5.** If  $(V_i, U_i)$  is a good system of representations of Example 3.1, then for any  $G$ -variety  $X$  the connecting maps  $X \times^G U_i \rightarrow X \times^G U_{i+1}$  factor as in Lemma 3.4, i.e., we have  $X \times^G U_i \rightarrow X \times^G (U_i \oplus W_i) \rightarrow X \times^G U_{i+1}$ .

**Example 3.6.** Let  $\mathbf{h}^G$  be the equivariant theory of Example 3.1. Then the property (CD) holds for  $\mathbf{h}^G$ .

Indeed, consider a good system of representations  $\{(V_j, U_j)\}_j$  for  $X$ . The subvarieties  $X_i \times^G U_j$ ,  $i = 0 \dots n$  form a cellular filtration on  $X \times^G U_j$  over  $S \times^G U_j$ . Note that  $\tilde{X}_i \times^G U_j$  is a resolution of singularities of  $X_i \times^G U_j$ . By (CD) for  $\mathbf{h}$  the set  $\{(f_i \times^G \text{id}_{U_j})_*(1)\}_i$  forms a basis of  $\mathbf{h}(X \times^G U_j)$  as a  $\mathbf{h}(S \times^G U_j)$ -module. By Lemma 3.4 the following diagram commutes:

$$\begin{array}{ccc} \mathbf{h}(\tilde{X}_i \times^G U_{j+1}) & \xrightarrow{(g_{i,j+1})_*} & \mathbf{h}(X \times^G U_{j+1}) \\ \downarrow \tilde{i}_j^* & & \downarrow i_j^* \\ \mathbf{h}(\tilde{X}_i \times^G U_j) & \xrightarrow{(g_{i,j})_*} & \mathbf{h}(X \times^G U_j) \end{array}$$

So  $i_m^*((f_i \times^G \text{id}_{U_{j+1}})_*(1)) = (f_i \times^G \text{id}_{U_j})_*(1)$ , which implies that the elements  $f_{i*}(1) = \lim_j ((f_i \times^G \text{id}_{U_j})_*(1))$  form a basis of  $\mathbf{h}^G(X)$  over  $\mathbf{h}^G(S)$ .

As for usual oriented theories we then obtain

**Lemma 3.7.** *Assume that  $\mathbf{h}^G$  satisfies the property (CD). Let  $X$  be a  $G$ -equivariant cellular space over  $S$ . Then the pairing  $(\cdot, \cdot): \mathbf{h}^G(X) \otimes_{\mathbf{h}^G(S)} \mathbf{h}^G(X) \rightarrow \mathbf{h}^G(S)$  given by  $(a, b) = p_*(ab)$  is non-degenerate and the map*

*$f: (\mathbf{h}^G(X \times_S X), \circ) \rightarrow \text{End}_{\mathbf{h}^G(S)} \mathbf{h}^G(X)$  given by  $a \mapsto f_a$ ,  $f_a(x) = (p_2)_*(p_1^*(x) \cdot a)$  is an  $\mathbf{h}^G(S)$ -linear isomorphism of rings. In particular, there is an  $\mathbf{h}^G(S)$ -linear isomorphism*

$$(\mathbf{h}_{\dim_S X}^G(X \times_S X), \circ) \rightarrow \text{End}_{\mathbf{h}^G\text{-}M(S)}(\mathbf{h}^G(X)),$$

*where  $\mathbf{h}^G\text{-}M(S)$  is the respective category of  $G$ -equivariant motives.*

#### 4. THE CONVOLUTION PRODUCT

In the present section we introduce the convolution product on the equivariant Borel-Moore homology (Definition 4.3) of the product  $G \times G \times \dots \times G$ . We relate this product to the usual correspondence product for the associated torsors (Lemma 4.6) and study its behaviour under the base change (diagram (6)).

Let  $G$  be a smooth algebraic group over  $k$  and let  $E$  be a  $G$ -torsor over  $k$  ( $G$  acts on the right). By definition there is an isomorphism  $\rho: E \times_k G \xrightarrow{\sim} E \times_k E$  given on points by  $(e, g) \mapsto (e, eg)$ . For each  $i \geq 0$  it induces an isomorphism

$$\rho_i: E \times_k G^i \longrightarrow E^{i+1}, \quad (e, g_1, g_2, \dots, g_i) \mapsto (e, eg_1, eg_2, \dots, eg_i).$$

Consider the composition

$$\gamma_i: E^{i+1} \xrightarrow{\rho_i^{-1}} E \times_k G^i = E \times_k G^i \xrightarrow{pr} G^i.$$

The coordinate-wise right  $G^{i+1}$ -action on  $E^{i+1}$  induces an action on  $E \times_k G^i$  and, hence, on  $G^i$ . For instance, on points it is given by

$$(3) \quad (e, g_1, \dots, g_i) \cdot (h_1, \dots, h_{i+1}) = (eh_1, h_1^{-1}g_1h_2, \dots, h_1^{-1}g_ih_{i+1}).$$

Consider projections  $p_j: E^{i+1} \rightarrow E^i$  obtained by removing the  $j$ -th coordinate and the respective  $G^i$ -action on  $E^i$ . For each  $i \geq 1$ ,  $1 \leq j \leq i+1$  there is a commutative diagram of  $G^i$ -equivariant maps

$$(4) \quad \begin{array}{ccc} E^{i+1} & \xrightarrow{\gamma_i} & G^i \\ p_j \downarrow & & \downarrow \pi_j \\ E^i & \xrightarrow{\gamma_{i-1}} & G^{i-1} \end{array}$$

where  $\pi_1(g_1, \dots, g_i) = (g_1^{-1}g_2, \dots, g_1^{-1}g_i)$  and  $\pi_j(g_1, \dots, g_i) = (g_1, \dots, \hat{g}_{j-1}, \dots, g_i)$  for  $j > 1$ .

**Example 4.1.** For  $i = 1$  it gives a commutative diagram of  $G$ -equivariant maps

$$\begin{array}{ccc} E \times_k E & \xrightarrow{\gamma_1} & G \\ p_j \downarrow & & \downarrow \pi_j \\ E & \xrightarrow{\gamma_0} & \text{Spec } k \end{array}$$

where  $\gamma_0, \pi_1, \pi_2$  are the structure maps,  $p_1, p_2$  are the corresponding projections and  $\gamma_1(e, eg) = g$ . Moreover, if  $E$  is trivial, then  $\gamma_1 = \pi_1: G \times_k G \rightarrow G$ ,  $(g_1, g_2) \mapsto g_1^{-1}g_2$ .

Let  $H$  be an algebraic subgroup of  $G$  such that  $G/H$  is a smooth variety over  $k$ . We can view  $G^i$  as an  $H$ -torsor over  $G^i/H$ , where  $H$  acts on  $G^i$  via the  $j$ th coordinate of  $G^{i+1}$ . By definition, the  $H^i$ -equivariant map  $\pi_j$  factors as

$$\pi_j: G^i \xrightarrow{q} G^i/H \xrightarrow{\bar{\pi}_j} G^{i-1},$$

where the second map  $\bar{\pi}_j$  is a fibration with a fibre  $G/H$ .

**Example 4.2.** The map  $\pi_1$  factors through the quotient maps modulo the diagonal action

$$\pi_1: G^i \xrightarrow{q} G^i/\Delta(H) \xrightarrow{\bar{\pi}_1} G^i/\Delta(G) = G^{i-1}.$$

which are equivariant with respect to the usual coordinate-wise  $H^i$ -action.

Consider an equivariant Borel-Moore homology theory  $\mathbf{h}$ . For every  $1 \leq j \leq i+1$  consider the action of the  $j$ -th copy of  $H$  on  $G^i$ . The property (Tor) gives an isomorphism

$$(5) \quad \mathbf{h}_{H^i}(G^i/H) \xrightarrow{\simeq} \mathbf{h}_{H^{i+1}}(G^i),$$

where  $H^{i+1}$  acts on  $G^i$  as in (3). Unless explicitly mentioned we will always identify these two rings.

Set  $\mathbf{S} = \mathbf{h}_H(G^0) = \mathbf{h}_H(k)$  and set the convolution product on  $\mathbf{S}$  to be the usual intersection product.

**Definition 4.3.** Assume that  $G/H$  is a smooth projective variety over  $k$ . We define the  $\mathbf{S}$ -linear convolution product  $'\circ'$  on  $\mathbf{h}_{H^i}(G^{i-1})$ ,  $i \geq 2$  to be the composite

$$\mathbf{h}_{H^i}(G^{i-1}) \otimes \mathbf{h}_{H^i}(G^{i-1}) \xrightarrow{\bar{\pi}_{i-1}^* \otimes \bar{\pi}_{i+1}^*} \mathbf{h}_{H^{i+1}}(G^i) \otimes \mathbf{h}_{H^{i+1}}(G^i) \xrightarrow{'\circ'}$$

$$\mathbf{h}_{H^{i+1}}(G^i) \xrightarrow{(\bar{\pi}_i)^*} \mathbf{h}_{H^i}(G^{i-1}),$$

where  $\mathbf{h}_{H^{i+1}}(G^i)$  is identified with  $\mathbf{h}_{H^i}(G^i/H)$  via (5) and  $\bar{\pi}_i$  is projective because so is  $G/H$ .

The central object of the present paper is the convolution ring  $(\mathbf{h}_{H^2}(G), \circ)$ , i.e., the case  $i = 2$ . In the next sections we will show that  $(\mathbf{h}_{B^2}(G), \circ)$  (where  $B$  is a Borel subgroup of a semisimple split  $G$ ) can be identified with the formal affine Demazure algebra.

**Example 4.4.** In the case  $i = 3$  the convolution ring  $(\mathbf{h}_{H^3}(G^2), \circ)$  is isomorphic to  $\mathbf{h}_{\Delta(H)}((G/H)^2)$  with respect to the usual correspondence product. Indeed, the maps  $\pi_i: G^3 \rightarrow G^2$ ,  $i = 2, 3, 4$  induce  $\Delta(H)$ -equivariant projections  $(G/H)^3 \rightarrow (G/H)^2$ . The isomorphism then follows by (Tor).

Observe that if  $G/H$  is an  $H$ -equivariant cellular space and  $\mathbf{h}_H$  satisfies (CD), then by Lemma 3.7 there is an  $\mathbf{S}$ -linear ring isomorphism

$$(\mathbf{h}_{H^3}(G^2), \circ) \simeq \text{End}_{\mathbf{S}} \mathbf{h}_H(G/H).$$

**Lemma 4.5.** *For  $i \geq 1$  the map  $\pi_1$  induces an injective ring homomorphism with respect to the convolution products*

$$(\mathbf{h}_{H^i}(G^{i-1}), \circ) \xrightarrow{\bar{\pi}_1^*} (\mathbf{h}_{H^{i+1}}(G^i), \circ).$$

*Proof.* For  $i = 1$  it follows from the fact that the convolution product on  $\mathbf{h}_{H^2}(G)$  is  $\mathbf{S}$ -linear.

For  $i \geq 2$  for each  $i - 1 \leq j \leq i + 1$  we have  $\pi_j \circ \pi_1 = \pi_1 \circ \pi_{j+1}$ . Since push-forwards commute with flat pull-backs by (TS), there are commutative diagrams in equivariant cohomology

$$\begin{array}{ccc} \mathbf{h}_{H^{i+1}}(G^i) & \xrightarrow{\bar{\pi}_1^*} & \mathbf{h}_{H^{i+2}}(G^{i+1}) \\ (\bar{\pi}_i)_* \downarrow \uparrow \bar{\pi}_{i-1}^*, \bar{\pi}_{i+1}^* & & (\bar{\pi}_{i+1})_* \downarrow \uparrow \bar{\pi}_i^*, \bar{\pi}_{i+2}^* \\ \mathbf{h}_{H^i}(G^{i-1}) & \xrightarrow{\bar{\pi}_1^*} & \mathbf{h}_{H^{i+1}}(G^i) \end{array}$$

Finally, there is a  $H^i$ -equivariant section of the map  $\bar{\pi}_1: G^i/\Delta(H) \rightarrow G^{i-1}$  given by  $(g_1, \dots, g_{i-1}) \mapsto (1, g_1, \dots, g_{i-1})$ , so  $\bar{\pi}_1^*$  is injective.  $\square$

**Lemma 4.6.** *The map  $\gamma_1$  induces a ring homomorphism*

$$(\mathbf{h}_{H^2}(G), \circ) \xrightarrow{\gamma_1^*} (\mathbf{h}_{H^2}(E^2), \circ) \xrightarrow{\simeq} (\mathbf{h}((E/H)^2), \circ),$$

where the last ring is viewed with respect to the correspondence product (2).

*Proof.* By (TS) the diagram (4) gives rise to commutative diagrams in cohomology

$$\begin{array}{ccc} \mathbf{h}_{H^3}(G^2) & \xrightarrow{\gamma_2^*} & \mathbf{h}_{H^3}(E^3) \\ (\bar{\pi}_2)_* \downarrow \uparrow \bar{\pi}_1^*, \bar{\pi}_3^* & & (p_2)_* \downarrow \uparrow p_1^*, p_3^* \\ \mathbf{h}_{H^2}(G) & \xrightarrow{\gamma_1^*} & \mathbf{h}_{H^2}(E^2) \end{array}$$

The last isomorphism follows by (Tor).  $\square$

Let  $\bar{k}$  denote the splitting field of a  $G$ -torsor  $E$  so that  $G_{\bar{k}} = E_{\bar{k}}$ . Since the base change preserves the convolution product, combining Lemmas 4.5 and 4.6 we obtain two commutative diagrams of convolution (correspondence) rings

$$\begin{array}{ccccc} \gamma_1^* : \mathfrak{h}_{H^2}(G) & \xrightarrow{pr^*} & \mathfrak{h}_{H^2}(E \times_k G) & \xrightarrow[\simeq]{\rho_1^*} & \mathfrak{h}_{H^2}(E^2) \\ \text{\scriptsize $res_{\bar{k}/k}$} \downarrow & & & & \downarrow \text{\scriptsize $res_{\bar{k}/k}$} \\ \gamma_1^* : \mathfrak{h}_{H^2}(G_{\bar{k}}) & \xrightarrow{\bar{\pi}_1^*} & \mathfrak{h}_{H^2}(G_{\bar{k}}^2/\Delta(H)) & \xrightarrow{q^*} & \mathfrak{h}_{H^2}(G_{\bar{k}}^2) \end{array}$$

and

$$\begin{array}{ccc} \gamma_0^* : \mathfrak{h}_H(k) & \xrightarrow{\rho_0^* \circ pr^*} & \mathfrak{h}_H(E) \\ \text{\scriptsize $res_{\bar{k}/k}$} \downarrow \simeq & & \downarrow \text{\scriptsize $res_{\bar{k}/k}$} \\ \gamma_0^* : \mathfrak{h}_H(\bar{k}) & \xrightarrow{q^* \circ \bar{\pi}_1^*} & \mathfrak{h}_H(G_{\bar{k}}) \end{array}$$

where  $res_{\bar{k}/k}$  is the base change map. Combining these two diagrams we obtain a commutative diagram of convolution rings

$$(6) \quad \begin{array}{ccc} \mathfrak{h}_H(E) \otimes_{\mathbf{S}} \mathfrak{h}_{H^2}(G) & \xrightarrow{(p_1^*, \gamma_1^*)} & \mathfrak{h}_{H^2}(E^2) \\ \text{\scriptsize $res_{\bar{k}/k}$} \downarrow & & \downarrow \text{\scriptsize $res_{\bar{k}/k}$} \\ \mathfrak{h}_H(G_{\bar{k}}) \otimes_{\mathbf{S}} \mathfrak{h}_{H^2}(G_{\bar{k}}) & \xrightarrow{(p_1^*, \gamma_1^*)} & \mathfrak{h}_{H^2}(G_{\bar{k}}^2), \end{array}$$

where the left convolution rings are  $\mathfrak{h}_H(E)$ - and  $\mathfrak{h}_H(G_{\bar{k}})$ -linear.

## 5. THE SUBRING OF PUSH-PULL OPERATORS

In the present section we prove that if  $H$  is the Borel subgroup of a split semisimple linear algebraic group, then the convolution ring  $\mathfrak{h}_{H^2}(G)$  of Definition 4.3 can be identified with the subring of push-pull operators (Corollary 5.3). Our arguments are essentially based on the Bruhat decomposition of  $G$  stated using the  $G$ -orbits on the product  $G/H \times_k G/H$  and the resolution of singularities (8).

As before assume that  $G/H$  is a smooth projective variety over  $k$ . In the notation of the previous section consider the  $H^2$ -equivariant maps of Example 4.2.

$$\pi_1 : G^2 \xrightarrow{q} G^2/\Delta(H) \xrightarrow{\bar{\pi}_1} G^2/\Delta(G) = G, \quad (g_1, g_2) \mapsto g_1^{-1}g_2.$$

Since  $G^2$  is a  $\Delta(G)$ -torsor over  $G$  ( $\Delta(H)$ -torsor over  $G^2/\Delta(H)$ ), by the property (Tor) the induced  $\Delta(G) \times H^2$ -equivariant pull-backs on cohomology coincide with the forgetful maps

$$(7) \quad \begin{array}{ccccc} \gamma_1^* : \mathfrak{h}_{H^2}(G) \simeq \mathfrak{h}_{\Delta(G) \times H^2}(G^2) & \xrightarrow{\bar{\pi}_1^*} & \mathfrak{h}_{\Delta(H) \times H^2}(G^2) & \xrightarrow{q^*} & \mathfrak{h}_{H^2}(G^2) \\ \simeq \downarrow & & \downarrow \simeq & & \downarrow \simeq \\ \mathfrak{h}_G((G/H)^2) & \xrightarrow{\quad} & \mathfrak{h}_H((G/H)^2) & \longrightarrow & \mathfrak{h}((G/H)^2) \end{array}$$

Moreover, by Lemma 4.5 it is a commutative diagram of convolution rings.

Let  $G$  be a split semisimple linear algebraic group over  $k$  and let  $\mathbf{h}$  be an equivariant theory that satisfies property (CD). We fix a Borel subgroup  $B$  of  $G$  containing a split maximal torus  $T$ . By Bruhat decomposition (e.g. [28])

$$G = \coprod_{w \in W} B \dot{w} B, \quad \dot{w} \in N_T,$$

is the disjoint union of  $B^2$ -orbits of  $G$ , where  $W = N_T/T$  is the Weyl group and  $N_T$  is the normalizer of  $T$  in  $G$ . Projecting this decomposition onto  $X = G/B$  gives a  $B$ -equivariant cellular filtration on  $X$  by closures  $\overline{X}_w$  of affine spaces  $X_w = B \dot{w} B/B$  of dimension  $l(w)$  (the length of  $w$ ). The preimage  $\pi_1^{-1}(B \dot{w} B)$  is a  $\Delta(G)$ -orbit in  $G^2$  (here  $H = B$ ). Let  $\mathcal{O}_w$  denote its image via  $G^2 \rightarrow X^2$  and let  $\overline{\mathcal{O}}_w$  denote its closure. Observe that both  $\mathcal{O}_w$  and  $\overline{\mathcal{O}}_w$  are  $\Delta(G)$ -invariant in  $X^2$ .

By properties of the Bruhat decomposition (see [28, §1]) it follows that the projection  $\mathcal{O}_w \rightarrow X^2 \rightarrow X$  is a torsor of a vector bundle over  $X$  with fibre  $X_w$ . Indeed, the transition functions are affine since they are given by the action of  $B$  on the left on  $B \dot{w} B/B$  that is by  $T$  acting on the product of the respective root subgroups  $\prod_{\alpha \in \Phi^+ \cap w(\Phi^-)} U_\alpha$  via the conjugation and, hence, by  $T$  acting on the product of the respective  $\mathbb{G}_a$ 's via the multiplication  $t \cdot x = \alpha(t)x$ ,  $t \in T$ ,  $x \in \mathbb{G}_a$ . So  $X^2$  is a  $G$ -equivariant ( $G$  acts diagonally) cellular space over  $X$  with filtration given by the closures  $\overline{\mathcal{O}}_w$ .

Assume that for each  $w \in W$  we are given a  $G$ -equivariant resolution of singularities  $\tilde{\mathcal{O}}_w \rightarrow \overline{\mathcal{O}}_w$ . Let  $[\tilde{\mathcal{O}}_w]_G$  denote the respective class in  $\mathbf{h}_G^{\dim_k X - l(w)}(X^2)$ . Then by the property (CD) the cohomology  $\mathbf{h}_G(X^2)$  (resp.  $\mathbf{h}_B(X^2)$  and  $\mathbf{h}(X^2)$ ) is a free module over  $\mathbf{h}_G(X)$  (resp. over  $\mathbf{h}_B(X)$  and  $\mathbf{h}(X)$ ) with basis  $\{[\tilde{\mathcal{O}}_w]_G\}_{w \in W}$  (resp.  $\{[\tilde{\mathcal{O}}_w]_B\}_{w \in W}$  and  $\{[\tilde{\mathcal{O}}_w]\}_{w \in W}$ ). Hence, the forgetful maps of (7) send  $[\tilde{\mathcal{O}}_w]_G \mapsto [\tilde{\mathcal{O}}_w]_B \mapsto [\tilde{\mathcal{O}}_w]$  and change the coefficients by  $-\otimes_{\mathbf{h}_G(X)} \mathbf{h}_B(X)$  and  $-\otimes_{\mathbf{h}_B(X)} \mathbf{h}(X)$  respectively, where the map  $\mathbf{S} = \mathbf{h}_G(X) \hookrightarrow \mathbf{h}_B(X) \rightarrow \mathbf{h}(X)$  is the classical characteristic map.

We now construct such  $G$ -equivariant resolutions as follows. For the  $i$ -th simple reflection  $s_i$  we denote  $X_{s_i}$  (resp.  $\mathcal{O}_{s_i}$ ) simply by  $X_i$  (resp. by  $\mathcal{O}_i$ ). Let  $P_i$  be the minimal parabolic subgroup corresponding to a simple root  $\alpha_i$  and let  $q_i: X \rightarrow G/P_i$  denote the respective quotient map.

**Lemma 5.1.** *We have  $\overline{\mathcal{O}}_i = X \times_{G/P_i} X$  and, in particular,  $\overline{\mathcal{O}}_i$  is smooth.*

*Proof.* We have  $(g_1 B, g_2 B) \in X \times_{G/P_i} X$ ,  $g_1, g_2 \in G$  if and only if  $g_1 P_i = g_2 P_i$ , so  $g_2 = g_1 h$  for some  $h \in P_i$ . Since  $P_i = B \cup B s_i B$ , it means that either  $g_2 B = g_1 B$  or  $g_2 B = g_1 B s_i B$ , so  $(g_1 B, g_2 B) \in \mathcal{O}_{s_i} \cup \Delta_X = \overline{\mathcal{O}}_i$ .  $\square$

For any  $w \in W$  we choose a reduced decomposition  $w = s_{i_1} s_{i_2} \dots s_{i_l}$  and set  $I_w = (i_1, i_2, \dots, i_l)$ . Consider a variety

$$(8) \quad \tilde{\mathcal{O}}_{I_w} = X \times_{G/P_{i_1}} X \times_{G/P_{i_2}} \dots \times_{G/P_{i_l}} X.$$

The projection on the first and the last factor  $pr: \tilde{\mathcal{O}}_{I_w} \rightarrow X \times_k X$  gives a  $G$ -equivariant resolution of singularities of  $\overline{\mathcal{O}}_w$ .

**Theorem 5.2.** *For  $H = B$  or 1, the image of  $[\tilde{\mathcal{O}}_{I_w}]_H \in \mathbf{h}_H(X \times_k X)$  under the Künneth isomorphism*

$$(\mathbf{h}_H(X \times_k X), \circ) \xrightarrow{\sim} \text{End}_{\mathbf{h}_H(k)}(\mathbf{h}_H(X))$$

is the composition of push-pull operators  $q_{i_1}^* q_{i_1*} \circ \dots \circ q_{i_l}^* q_{i_l*}$ .

*Proof.* By definition the image of  $[\tilde{\mathcal{O}}_{I_w}]_H$  is the  $\mathbf{h}_H(k)$ -linear operator

$$\mathbf{h}_H^\bullet(X) \xrightarrow{p_1^*} \mathbf{h}_H(X \times_k X) \xrightarrow{[\tilde{\mathcal{O}}_{I_w}]} \mathbf{h}_H(X \times_k X) \xrightarrow{p_{2*}} \mathbf{h}_H^{\bullet-l(w)}(X).$$

By the projection formula and (TS) it can be also written as

$$\mathbf{h}_H^\bullet(X) \xrightarrow{pr_{i+1}^*} \mathbf{h}_H(\tilde{\mathcal{O}}_{I_w}) \xrightarrow{pr_1^*} \mathbf{h}_H^{\bullet-l(w)}(X),$$

where  $pr_j$  denotes the projection on the  $j$ -th coordinate (recall that  $p_j$  denotes the projection obtained by removing the  $j$ -th coordinate).

By the property (TS) we obtain a commutative diagram

$$\begin{array}{ccccccc} \mathbf{h}_H(X) & \xrightarrow{pr_2^*} & \mathbf{h}_H(\tilde{\mathcal{O}}_{i_l}) & \xrightarrow{pr_{23}^*} & \mathbf{h}_H(\tilde{\mathcal{O}}_{(i_{l-1}, i_l)}) & \xrightarrow{pr_{234}^*} & \dots \longrightarrow \mathbf{h}_H(\tilde{\mathcal{O}}_{I_w}) \\ q_{i_l*} \downarrow & & pr_{1*} \downarrow & & pr_{12*} \downarrow & & \downarrow \\ \mathbf{h}_H(G/P_{i_l}) & \xrightarrow{q_{i_l}^*} & \mathbf{h}_H(X) & \xrightarrow{pr_2^*} & \mathbf{h}_H(\tilde{\mathcal{O}}_{i_{l-1}}) & & \dots \\ & & q_{i_{l-1}*} \downarrow & & pr_{1*} \downarrow & & \downarrow \\ & & \mathbf{h}_H(G/P_{i_{l-1}}) & \xrightarrow{q_{i_{l-1}}^*} & \mathbf{h}_H(X) & & \dots \\ & & & & q_{i_{l-2}*} \downarrow & & \downarrow \\ & & & & \dots & \longrightarrow & \dots \longrightarrow \mathbf{h}_H(X) \end{array}$$

where  $pr_{ijk\dots}$  denote the projection on the  $i$ -th,  $j$ -th,  $k$ -th,  $\dots$ , coordinates. The result then follows since the top horizontal row gives  $pr_{i+1}^*$  and the right vertical column gives  $pr_{1*}$ .  $\square$

Observe that the theorem can not be stated for  $H = G$  as  $X$  is not a  $G$ -equivariant cellular space so we can not use the Künneth isomorphism of Lemma 3.7.

Combining Diagram (7) and Theorem 5.2 we obtain

**Corollary 5.3.** *There is a commutative diagram of convolution rings*

$$\begin{array}{ccccccc} \mathbf{h}_{B^2}(G) & \xrightarrow{\pi_1^*} & \mathbf{h}_{\Delta(B) \times B^2}(G^2) & \xrightarrow{\simeq} & \mathbf{h}_B(X^2) & \xrightarrow{\simeq} & \text{End}_{\mathbf{S}}(\mathbf{h}_B(X)) \\ & & q^* \downarrow & & \downarrow & & \downarrow \\ & & \mathbf{h}_{B^2}(G^2) & \xrightarrow{\simeq} & \mathbf{h}(X^2) & \xrightarrow{\simeq} & \text{End}_{\mathbf{R}}(\mathbf{h}(X)) \end{array}$$

where the image of  $(\mathbf{h}_{B^2}(G), \circ)$  in  $\text{End}_{\mathbf{S}}(\mathbf{h}_B(X))$  is the subring generated by the push-pull operators  $q_i^* q_{i*}$  (of degree  $(-1)$ ) and the image of the forgetful map  $\mathbf{S} = \mathbf{h}_G^\bullet(X) \rightarrow \mathbf{h}_B^\bullet(X)$  (of degrees  $\bullet$ ) and the last vertical arrow is induced by the augmentation map  $\mathbf{S} \rightarrow \mathbf{R} = \mathbf{h}(k)$ .

## 6. SELF-DUALITY OF THE ALGEBRA OF PUSH-PULL OPERATORS

In the present section we identify the convolution ring  $\mathbf{h}_{B^2}(G)$  with the formal affine Demazure algebra  $\mathbf{D}_F$  of [19] and show that it is self-dual with respect to the convolution product (Theorem 6.2). Our arguments are based on the results of [19], [7], [8] and, especially, [9]. We use the notation of [9].

Recall that algebraic oriented cohomology theories  $\mathbf{h}$  correspond (up to universality) to one-dimensional commutative formal group laws  $F(u, v)$ : the formal group law corresponds to  $\mathbf{h}$  by means of the Quillen formula expressing the first characteristic classes

$$c_1^{\mathbf{h}}(\mathcal{L}_1 \otimes \mathcal{L}_2) = F(c_1^{\mathbf{h}}(\mathcal{L}_1), c_1^{\mathbf{h}}(\mathcal{L}_2))$$

and the respective cohomology theory  $\mathbf{h}$  is defined from  $F$  by tensoring with the algebraic cobordism

$$\mathbf{h}(-) = \Omega(-) \otimes_{\Omega(k)} \mathbf{R},$$

where  $\Omega(k) \rightarrow \mathbf{R}$  defines  $F$  by specializing the coefficients in the Lazard ring (see [9, §2] for details). For example, the additive formal group law correspond to Chow groups and the periodic multiplicative law corresponds to  $K$ -theory.

By [9, Thm. 3.3] the completed  $B$ -equivariant coefficient ring  $\mathbf{S} = \mathbf{h}_B(k)$  can be identified with the formal group algebra  $\mathbf{R}[[T^*]]_F$ , where  $T^*$  is the group of characters of a split maximal torus  $T \subset B$  and  $F$  is the respective formal group law.

Following [9, §5] (we assume that  $\mathbf{S}$  satisfies regularity condition [9, 5.1]) consider the localized algebra  $\mathbf{Q} = \mathbf{R}[[T^*]]_F[\frac{1}{x_\alpha}]_\alpha$  (where  $\alpha$  runs through all simple roots) and the smash products  $\mathbf{Q}_W = \mathbf{Q} \otimes_{\mathbf{R}} \mathbf{R}[W]$  and  $\mathbf{S}_W = \mathbf{S} \otimes_{\mathbf{R}} \mathbf{R}[W]$  with the multiplication given by

$$q\delta_w \cdot q'\delta_{w'} = q(wq')\delta_{ww'}$$

for  $q, q' \in \mathbf{Q}$  (respectively  $\mathbf{S}$ ) and  $w, w' \in W$  (the Weyl group). Consider the duals  $\mathbf{Q}_W^* = \text{Hom}_{\mathbf{Q}}(\mathbf{Q}_W, \mathbf{Q})$  and  $\mathbf{S}_W^* = \text{Hom}_{\mathbf{S}}(\mathbf{S}_W, \mathbf{S})$ . By definition  $\mathbf{Q}_W^*$  and  $\mathbf{S}_W^*$  can be identified with the ring of functions  $\text{Hom}(W, \mathbf{Q})$  and  $\text{Hom}(W, \mathbf{S})$  respectively

As in [19, Def. 6.2, 6.3] for each simple root  $\alpha_i$  of the root system for  $G$  define the push-pull element

$$Y_i = (1 + \delta_i) \frac{1}{x_{-i}} \in \mathbf{Q}_W.$$

Define the formal affine Demazure algebra  $\mathbf{D}_F$  as the subalgebra of  $\mathbf{Q}_W$  generated by multiplications by  $\mathbf{S}$  and the elements  $Y_i$ .

By [7, Thm. 7.9] (see also [19, Thm. 5.14]) the  $\mathbf{R}$ -algebra  $\mathbf{D}_F$  satisfies the following (complete) set of relations: for  $i, j = 1 \dots rk(G)$  and  $u \in \mathbf{S}$

- $Y_i^2 = \kappa_i Y_i$ , where  $\kappa_i = \frac{1}{x_i} + \frac{1}{x_{-i}}$  and  $x_i = x_{\alpha_i}$ ,
- $Y_i u = s_i(u) Y_i + \Delta_{-i}(u)$ , where  $\Delta_{-i}(u) = \frac{u - s_i(u)}{x_{-i}}$ ,
- $(Y_i Y_j)^{m_{ij}} - (Y_j Y_i)^{m_{ij}} = \sum_{I_w} c_{I_w} Y_{I_w}$ , where the sum is taken over all reduced expressions  $I_w$  of elements  $w$  of the subgroup  $\langle s_i, s_j \rangle \subseteq W$ , and the coefficients  $c_{I_w}$  are given by the formulas of [19, Prop. 5.8]

**Example 6.1.** If  $F$  corresponds to Chow groups, then  $\mathbf{D}_F = \mathbf{H}_{nil}$  is the affine nil-Hecke algebra over  $\mathbb{Z}$  in the notation of [17]. If  $F$  corresponds to  $K$ -theory, then  $\mathbf{D}_F$  is the 0-affine Hecke algebra over  $\mathbb{Z}$  ( $q \rightarrow 0$  in the affine Hecke algebra). If  $F$  corresponds to the generic hyperbolic formal group law of [8, §9], then by [8, Prop. 9.2] the constant part of  $\mathbf{D}_F$  is isomorphic to the localized classical Iwahori-Hecke algebra.

Let  $\mathbf{D}_F^* = \text{Hom}_{\mathbf{S}}(\mathbf{D}_F, \mathbf{S})$  denote its dual. Observe that the main result of [9] (Thm. 8.2 loc.cit.) says that  $\mathbf{D}_F^*$  is isomorphic to the  $\mathbf{R}$ -algebra  $\mathbf{h}_T(X)$ . We then obtain the following generalization of [17, Prop. 12.8]

**Theorem 6.2.** *Let  $G$  be a split semisimple linear algebraic group over a field  $k$  and let  $\mathbf{h}$  be an equivariant theory that satisfies property (CD).*

*Then the convolution algebra  $(\mathbf{h}_{B^2}(G), \circ)$  is isomorphic (as an  $\mathbf{R}$ -algebra) to the formal affine Demazure algebra  $\mathbf{D}_F$ . So there is an  $\mathbf{R}$ -algebra isomorphism*

$$(\mathbf{D}_F^*, \circ) \simeq (\mathbf{D}_F, \cdot)$$

*Proof.* By Corollary 5.3 the ring  $(\mathbf{h}_{B^2}(G), \circ) \simeq (\mathbf{h}_B(X), \circ)$  is isomorphic to the subalgebra of  $\text{End}_{\mathbf{S}}(\mathbf{h}_B(X))$  generated by the image of the forgetful map  $\mathbf{h}_G(X) \rightarrow \mathbf{h}_B(X)$  and push-pull operators  $q_i^* q_{i*}$ . Since the map  $B \rightarrow B/T$  is an affine fibration, the natural map  $\mathbf{h}_B(X) \rightarrow \mathbf{h}_T(X)$  is an isomorphism. Hence we may identify  $\mathbf{S}$  with  $\mathbf{h}_T(k)$  and  $\text{End}_{\mathbf{S}}(\mathbf{h}_B(X))$  with  $\text{End}_{\mathbf{S}}(\mathbf{h}_T(X))$ . Observe that these identifications preserve push-pull operators. The inclusion of  $T$ -fixed point set  $W \rightarrow X$  gives an embedding  $\mathbf{h}_T(X) \rightarrow \mathbf{h}_T(W) = \mathbf{S}_W^* \subseteq \mathbf{Q}_W^*$ . By [9, Corollary 8.7] there is the following commutative diagram

$$(9) \quad \begin{array}{ccccc} \mathbf{h}_T(X) & \longrightarrow & \mathbf{S}_W^* & \hookrightarrow & \mathbf{Q}_W^* \\ q_i^* q_{i*} \downarrow & & & & \downarrow A_i \\ \mathbf{h}_T(X) & \longrightarrow & \mathbf{S}_W^* & \hookrightarrow & \mathbf{Q}_W^* \end{array}$$

where the Hecke operator  $A_i$  is given by

$$A_i(f)(x) = f(x \cdot Y_i) \quad \text{for } x \in \mathbf{Q}_W, f \in \mathbf{Q}_W^*.$$

Moreover, the forgetful map

$$\mathbf{S} \cong \mathbf{h}_G(X) \rightarrow \mathbf{h}_T(X) = \oplus_{w \in W} \mathbf{S}$$

is given by the formula  $s \mapsto (w \cdot s)_{w \in W}$  for any  $s \in \mathbf{S}$ . Then the multiplication in  $\mathbf{h}_T(X) = \mathbf{S}_W^*$  by the image of any element in  $s \in \mathbf{h}_G(X)$  induces a right multiplication by  $s$  in  $\mathbf{Q}_W^*$ . Since  $\mathbf{Q}_W$  is a free  $\mathbf{Q}$ -module of finite rank, the natural map  $\iota: \mathbf{Q}_W \rightarrow \text{End}_{\mathbf{Q}}(\mathbf{Q}_W^*)$  given by  $\iota(x)(f)(y) = f(yx)$  is an inclusion. Note that every  $A_i$  lies in the image of  $\iota$ . Then by diagram (9) the image of  $\mathbf{h}_{B^2}(G)$  is isomorphic to a subalgebra of  $\mathbf{Q}_W$  generated by  $\mathbf{S}$  and  $Y_i$  which is  $\mathbf{D}_F$ .  $\square$

## 7. THE RATIONAL ALGEBRA OF PUSH-PULL OPERATORS

In the present section we introduce the rational algebra of push-pull operators  $\overline{\mathbf{D}}_F$  (Definition 7.5) and show that it can be identified with the subring of rational endomorphisms of  $G/B$  (Theorem 7.6).

The  $B^2$ -equivariant isomorphism  $E \times_k G \rightarrow E \times_k E$ ,  $(e, g) \mapsto (e, eg)$  induces an isomorphism  $E \times^B G/B \rightarrow E/B \times_k E/B$ . For all  $w \in W$  fix a reduced decomposition  $I_w = (i_1, \dots, i_l)$  and the corresponding Bott-Samelson resolution  $X_{I_w} \rightarrow G/B$  of the Schubert cell. This map is  $B$ -equivariant, so it descends to a map  $Y_{I_w} = E \times^B X_{I_w} \rightarrow E \times^B G/B$ .

**Lemma 7.1.** *The classes  $[Y_{I_w}]$  form a basis of  $\mathbf{h}(E/B \times_k E/B)$  over  $\mathbf{h}(E/B)$ , where the module structure is given by the pullback of the projection  $pr_1^*: \mathbf{h}(E/B) \rightarrow \mathbf{h}(E/B \times_k E/B)$ .*

*Proof.* Since  $B$  is special,  $G$ -torsor  $E$  splits over the function field of  $E/B$ . Then by [25, Lemma 3.3] projection  $pr_1: E/B \times_k E/B \rightarrow E/B$  is a cellular fibration in



the sense of [25, Definition 3.1] so that  $(E/B)^2$  is a cellular space over  $E/B$ . Let  $\xi$  be the generic point of  $E/B$ . The pullback of an open embedding

$$j^* : \mathbf{h}(E/B \times_k E/B) \rightarrow \mathbf{h}(\xi \times_k E/B) \simeq \mathbf{h}(G/B)$$

is surjective and any preimage of  $\mathbf{R}$ -basis of  $\mathbf{h}(G/B)$  gives a basis of  $\mathbf{h}(E/B \times_k E/B)$ . Thus it is sufficient to check that  $j^*$  sends  $[Y_{I_w}]$  to a basis of  $\mathbf{h}(\xi \times_k E/B)$ . Let  $p : E \rightarrow E/B$  be the projection. Note that

$$E \times^B X_{I_w} \times_{(E/B \times_k E/B)} \xi \times_k E/B = p^{-1}(\xi) \times^B X_{I_w} = \xi \times X_{I_w},$$

since  $p^{-1}(\xi) \rightarrow \xi$  is a trivial  $B$ -torsor. Thus  $j^*([Y_{I_w}]) = [\xi \times X_{I_w}]$  forms a basis of  $\mathbf{h}(\xi \times E/B) = \mathbf{h}(\xi \times G/B)$  over  $\mathbf{h}(\xi) = \mathbf{R}$ .  $\square$

Consider a  $B$ -equivariant map

$$f : E \times^B G \rightarrow B \backslash G, \quad (e, g)B \mapsto Bg.$$

Let  $X'_{I_w} = (P_{i_1} \times \dots \times P_{i_l})/B^l$  where  $B^l$ -action on  $P_{i_1} \times \dots \times P_{i_l}$  is given by  $(p_1, \dots, p_l) \cdot (b_1, \dots, b_l) = (b_1^{-1}p_1b_2, \dots, b_l^{-1}p_l)$ . Then  $X'_{I_w}$  gives the Bott-Samelson class for  $B \backslash G$ .

**Lemma 7.2.** *The composition  $\mathbf{h}_B(B \backslash G) \xrightarrow{f^*} \mathbf{h}_B(E \times^B G) \simeq \mathbf{h}(E/B \times_k E/B)$  maps  $[X'_{I_w}]_B$  to  $[Y_{I_w}]$ .*

*Proof.* Consider the map  $P_{i_1} \times^B P_{i_2} \times^B \dots \times^B P_{i_l} \rightarrow G$  given by  $(p_1, \dots, p_l) \rightarrow p_1 \dots p_l$ . It is  $B$ -equivariant with respect to the left multiplication, so it descends to a map  $M_{I_w} = E \times^B P_{i_1} \times^B P_{i_2} \times^B \dots \times^B P_{i_l} \rightarrow E \times^B G$ . By construction we have an isomorphism

$$M_{I_w} \simeq Y_{I_w} \times_{E \times^B (G/B)} (E \times^B G).$$

Then  $[M_{I_w}]_B$  is mapped to  $[Y_{I_w}]$  via the isomorphism  $\mathbf{h}(E \times^B G/B) \rightarrow \mathbf{h}_B(E \times^B G)$ . Thus it is sufficient to check that  $f^*[X'_{I_w}]_B = [M_{I_w}]_B$ , which follows from the fact that

$$M_{I_w} = E \times^B (P_{i_1} \times \dots \times P_{i_l}/B^{l-1}) \simeq (E \times^B G) \times_{B \backslash G} X'_{I_w}. \quad \square$$

**Lemma 7.3.** (cf. [25, Corollary 3.4]) *The composition*

$$(p_1^*, \gamma_1^*) : \mathbf{h}_B(E) \otimes_{\mathbf{S}} \mathbf{h}_{B^2}(G) \longrightarrow \mathbf{h}_{B^2}(E^2) \simeq \mathbf{h}((E/B)^2)$$

*of the diagram (6) (for  $H = B$ ) is an isomorphism.*

*Proof.* Consider the basis of  $\mathbf{h}_{B^2}(G)$  over  $\mathbf{S}$  given by the classes of Bott-Samelson resolutions  $\zeta_{I_w}$ . Then by Lemma 7.2  $\gamma_1^*(\zeta_{I_w})$  forms a basis of  $\mathbf{h}_{B^2}(E^2)$  over  $\mathbf{h}_B(E)$  induced by the respective cellular filtration.  $\square$

Consider the restriction map  $\mathbf{h}(E/B) \rightarrow \mathbf{h}(E_{\bar{k}}/B) = \mathbf{h}(X_{\bar{k}})$  on cohomology induced by the scalar extension  $\bar{k}/k$  (here  $\bar{k}$  is a splitting field of  $E$ ). Let  $\bar{\mathbf{h}}(X)$  denote its image.

**Corollary 7.4.** *The image of the ring homomorphism*

$$\text{res}_{\bar{k}/k} : (\mathbf{h}(E/B \times_k E/B), \circ) \longrightarrow (\mathbf{h}(X_{\bar{k}} \times_{\bar{k}} X_{\bar{k}}), \circ).$$

*is the subalgebra generated by the multiplication by the elements of  $\bar{\mathbf{h}}(X)$  and the push-pull operators  $q_i^* q_{i*} : \mathbf{h}(X) \rightarrow \mathbf{h}(G/P_i) \rightarrow \mathbf{h}(X)$  for all simple roots  $\alpha_i$ .*

*Proof.* Follows by (6), Lemma 7.3 and Corollary 5.3.  $\square$

There is a natural action of  $W$  on  $\bar{\mathbf{h}}(X)$  that comes from the  $W$ -action on  $E/T$ . So we can endow  $\bar{\mathbf{h}}(X) \otimes_{\mathbf{S}} \mathbf{Q}_W$  with a structure of an  $\mathbf{R}$ -algebra.

**Definition 7.5.** Let  $\bar{\mathbf{D}}_F$  denote its subalgebra  $\bar{\mathbf{h}}(X) \otimes_{\mathbf{S}} \mathbf{D}_F$ . We call it the rational algebra of push-pull operators. By  $\bar{\mathbf{D}}_F^{(m)}$  we denote its degree  $m$  homogeneous component assuming that all  $Y_i$ 's have degree  $(-1)$  and elements of  $\bar{\mathbf{h}}^\bullet(X)$  (and of  $\mathbf{S} = \mathbf{h}_B^\bullet(k)$ ) have degree ' $\bullet$ '.

Observe that if  $E$  is split, then  $\bar{\mathbf{D}}_F = \mathbf{h}(X) \otimes_{\mathbf{S}} \mathbf{D}_F$  does not coincide with  $\mathbf{D}_F$ . Set  $N = \dim X$ .

**Theorem 7.6.** *Consider the restriction*

$$\text{res}_{\bar{k}/k}: \text{End}_{\mathbf{h}\text{-}M(k)}([E/B]) \longrightarrow \text{End}_{\mathbf{h}\text{-}M(\bar{k})}([X_{\bar{k}}])$$

*on endomorphism rings of the respective motives (i.e., preserving the grading of  $\mathbf{h}(X)$ ). Its image can be identified with  $\bar{\mathbf{D}}_F^{(0)}$  via the injective forgetful map*

$$\phi: (\bar{\mathbf{h}}(X) \otimes_{\mathbf{S}} \mathbf{h}_G(X_k^2))^{(N)}, \circ \longrightarrow (\mathbf{h}^N(X_k^2), \circ).$$

*Proof.* By (7) both  $\mathbf{h}_G(X^2)$  and  $\mathbf{h}(X^2)$  are free modules over  $\mathbf{h}_G(X)$  and  $\mathbf{h}(X)$  with basis given by the classes  $[\tilde{\mathcal{O}}_{I_w}]_G$  and  $[\tilde{\mathcal{O}}_{I_w}]$  respectively. The map  $\phi$  sends  $[\tilde{\mathcal{O}}_{I_w}]_G \mapsto [\tilde{\mathcal{O}}_{I_w}]$  and leaves the coefficients invariant. The result follows by Corollary 7.4, Corollary 5.3 and Theorem 6.2.  $\square$

We say that a (co-)homology theory  $\mathbf{h}$  satisfies the Dimension Axiom if

(Dim) For any smooth variety  $Y$  over  $k$  we have  $\mathbf{h}^n(Y) = 0$  for all  $n > \dim Y$ .

**Example 7.7.** Any theory  $\mathbf{h}$  over a field  $k$  of characteristic 0 obtained by specialization of coefficients of the Lazard ring (e.g. Chow groups, connective  $K$ -theory, algebraic cobordism  $\Omega$ ) satisfies (Dim). The graded  $K$ -theory  $K_0(-)[\beta, \beta^{-1}]$  of [24, Example 1.1.5] does not satisfy (Dim).

Observe that the image of the characteristic map  $c: \mathbf{S} \rightarrow \mathbf{h}(X)$  is contained in  $\bar{\mathbf{h}}(X)$  (see [16, Thm. 4.5]). Consider both the induced map  $\mathbf{c}: \mathbf{D}_F \rightarrow \bar{\mathbf{D}}_F$  and the restriction map  $\text{res}_{\bar{k}/k}: (\mathbf{h}_B(E) \otimes_{\mathbf{S}} \mathbf{D}_F) \rightarrow \bar{\mathbf{D}}_F$ . We will use the following substitute of the Rost nilpotence theorem.

**Lemma 7.8.** *Assume that the theory  $\mathbf{h}$  satisfies (Dim), then the kernels of  $\mathbf{c}$  and  $\text{res}_{\bar{k}/k}$  are complete. In other words, there is a commutative diagram of maps of convolution rings*

$$\begin{array}{ccc} \mathbf{D}_F & \xrightarrow{\gamma_1^*} & \mathbf{h}((E/B)^2) \\ & \searrow \mathbf{c} & \downarrow \text{res}_{\bar{k}/k} \\ & & \bar{\mathbf{D}}_F \end{array}$$

*with complete kernels (cf. [32, Ch.2, Lemma 2.2]).*

*Proof.* If  $\mathbf{c}(x) = 0$  (resp.  $\text{res}_{\bar{k}/k}(x) = 0$ ) then  $x = \sum_w a_w Y_{I_w}$  with  $a_w \in \mathbf{S}$ ,  $c(a_w) = 0$  (resp. with  $a_w \in \mathbf{h}(E/B)$ ,  $\text{res}_{\bar{k}/k}(a_w) = 0$ ). Since  $c$  (resp.  $\text{res}_{\bar{k}/k}$ ) commutes with push-pull operators on  $\mathbf{S}$  (resp. on  $\mathbf{h}(E/B)$ ), the product of  $n$  such elements  $x_1 \circ x_2 \circ \dots \circ x_n$  corresponds to

$$\left( \sum_w a_w^1 Y_{I_w} \right) \dots \left( \sum_w a_w^n Y_{I_w} \right) = \sum_w a_{w,n} Y_{I_w}, \quad a_{w,n} \in (\ker c)^n, \text{ (resp. } (\ker \text{res}_{\bar{k}/k})^n)$$

Since  $\ker c$  is contained in the augmentation ideal, and  $\mathbf{S}$  is complete, then the series  $\sum_n a_{w,n}$  converges in  $\mathbf{S}$  (resp.  $\ker \text{res}_{\bar{k}/k}$  is contained in  $\mathfrak{h}^{>0}(E/B)$ ), then  $a_{w,n} = 0$  for  $n > \dim E/B$ , thus  $\ker \mathfrak{c}$  is complete and  $\ker \text{res}_{\bar{k}/k}$  is nilpotent, hence complete.  $\square$

**Lemma 7.9.** *If  $E$  is a generic  $G$ -torsor in the sense of [15, §3, p.108], then the map  $\gamma_1^*$  and, hence,  $\mathfrak{c}$ , of the lemma 7.8 is surjective.*

*Proof.* Observe that if  $E$  is generic, then it admits a generically open  $G$ -equivariant embedding into  $\mathbb{A}_k^N$ . So the projection  $E \times_k G^i \rightarrow G^i$  in the definition of  $\gamma_i$  factors through  $\mathbb{A}_k^N \times_k G^i$ . By (Loc) and (HI) the induced pullback  $\gamma_i^*$  is surjective.  $\square$

## 8. APPLICATIONS AND EXAMPLES

Our main application is the following

**Theorem 8.1.** *Let  $G$  be a split semisimple linear algebraic group over a field  $k_0$  and let  $E$  be a  $G$ -torsor over a field extension  $k$  of  $k_0$ . Let  $\mathfrak{h}$  be an oriented cohomology theory over  $k$  that satisfies both (CD) and (Dim) axioms. Let  $\langle [E/B] \rangle_{\mathfrak{h}}$  (resp.  $\langle [E/B]_* \rangle_{\mathfrak{h}}$ ) denote the pseudo-abelian subcategory generated by the (resp. non-graded)  $\mathfrak{h}$ -motive of  $E/B$ . Then*

- (i) *There are isomorphisms between the Grothendieck groups*

$$K_0(\langle [E/B] \rangle_{\mathfrak{h}}) \simeq K_0(\overline{\mathbf{D}}_F^{(0)}) \quad \text{and} \quad K_0(\langle [E/B]_* \rangle_{\mathfrak{h}}) \simeq K_0(\overline{\mathbf{D}}_F).$$

- (ii) *There is a 1-1 correspondence between direct sum decompositions of the  $\mathfrak{h}$ -motive  $[E/B]$  and direct sum decompositions of the  $\overline{\mathbf{D}}_F^{(0)}$ -module  $\overline{\mathbf{D}}_F^{(0)}$ . Moreover, any two direct summands in the motivic decomposition of  $E/B$  that are Tate twists of each other correspond to isomorphic  $\overline{\mathbf{D}}_F$ -modules.*
- (iii) *If  $E$  is generic, then the algebra  $\overline{\mathbf{D}}_F$  in (i) and (ii) can be replaced by the algebra  $\mathbf{D}_F$ .*

*Proof.* Consider a graded endomorphism ring of the  $\mathfrak{h}$ -motive of  $E/B$

$$C_F = (\text{End}_{\mathfrak{h}\text{-}M(k)}^\bullet([E/B]), \circ).$$

Our main result (Theorem 7.6) together with Lemma 7.8 says that the restriction map gives a surjective ring homomorphism with complete kernel

$$\text{res}_{\bar{k}/k}: C_F \rightarrow \overline{\mathbf{D}}_F.$$

The part (i) then follows from [32, Ch. 2, Lemma 2.2].

The part (ii) follows from [25, §2 and Prop. 2.6] applied to the map  $\text{res}_{\bar{k}/k}$  (an isomorphism between Tate twists in the non-graded category of motives corresponds to an isomorphism between idempotents of [25, §2.1]).

The part (iii) follows from Lemma 7.8 applied to the map  $\mathfrak{c}$ .  $\square$

**Example 8.2.** If  $G$  is special, i.e.  $G$  is simple simply-connected of type  $A$  or  $C$ , then any  $G$ -torsor  $E$  splits, and the respective non-graded subcategory  $\langle [E/B]_* \rangle_{\mathfrak{h}}$  is generated by the motive of a point. So by (ii) and (iii) of the theorem all indecomposable direct summands of  $\overline{\mathbf{D}}_F$  (resp. of  $\mathbf{D}_F$ ) are isomorphic to the  $\overline{\mathbf{D}}_F$ - (resp.  $\mathbf{D}_F$ -)module  $\mathbf{S}$  and

$$K_0(\overline{\mathbf{D}}_F) \simeq K_0(\mathbf{D}_F) \simeq \mathbb{Z}.$$

**Lemma 8.3.** *If the coefficient ring  $\mathbf{R}$  is Artinian, then both  $C_F$  and  $\overline{\mathbf{D}}_F$  and, hence, the categories  $\langle [E/B]_* \rangle_{\mathbf{h}}$  and  $\text{Proj } \overline{\mathbf{D}}_F$  satisfy the Krull-Schmidt property (uniqueness of a direct sum decomposition).*

Observe that in general the ring  $\overline{\mathbf{D}}_F$  is not Krull-Schmidt (and not semi-simple).

*Proof.* If  $\mathbf{R}$  is Artinian, then both  $\overline{\mathbf{D}}_F$  and  $C_F$  are Artinian (as  $\overline{\mathbf{D}}_F$  is finite dimensional over  $\mathbf{R}$ ). So they are both Noetherian which implies that the respective tautological modules  $\overline{\mathbf{D}}_F$  and  $C_F$  have finite length and, hence, the Krull-Schmidt property holds for both  $\overline{\mathbf{D}}_F$  and  $C_F$ .  $\square$

As a direct application of the main result of [25] one obtains the following characterization of modular representations of the (affine) nil-Hecke algebra ( $F$  is an additive formal group law and  $\mathbf{h}(-) = CH(-; \mathbb{F}_p)$ ).

**Corollary 8.4.** *Let  $G$  be a split semisimple linear algebraic group over a field  $k_0$ . Consider the affine nil-Hecke algebra  $\mathbf{H}_{\text{nil},p}$  for  $G$  with coefficients in  $\mathbf{R} = \mathbb{F}_p$ ,  $p$  is a prime. Then*

$$K_0(\mathbf{H}_{\text{nil},p}) \simeq K_0(\langle \mathcal{R}_* \rangle),$$

where  $\mathcal{R}_*$  is the generalized (non-graded) Rost-Voevodsky motive corresponding to a generic  $G$ -torsor  $E$  and the prime  $p$ .

In particular, all indecomposable graded submodules of  $\mathbf{H}_{\text{nil},p}$  are isomorphic to a graded indecomposable submodule  $P_*$  corresponding to  $\mathcal{R}_*$ . Moreover, they are free  $\mathbf{S}$ -modules of rank  $r$  that is equal to the  $p$ -part of the product of  $p$ -exceptional degrees of  $G$  and we have

$$\mathbf{H}_{\text{nil},p} \simeq \text{Mat}_{|W|/r}(\text{End}(P_*)).$$

*Proof.* The  $\mathbf{S}$ -rank coincides with the number of Tate motives in the decomposition of  $\mathcal{R}$  over a splitting field of  $E$ , that is  $g_p(1) = \prod_{i=1}^r \frac{1-t^{d_i}p^{k_i}}{1-t^{d_i}}|_{t=1}$  (in the notation of [25]) which is equal to the  $p$ -part  $p^{\sum_{i=1}^r k_i}$  of  $p$ -exceptional degrees of [20, p.73]. The last statement follows from the fact that the ring of graded endomorphisms of  $\mathcal{R}_*$  coincides with the ring of endomorphisms of  $\mathcal{R}$ .  $\square$

We now switch to integer coefficients. Recall that in this case the Krull-Schmidt property usually fails.

**Example 8.5.** Consider the root system of type  $A_1$ . In this case  $T^* = \mathbb{Z}\omega$  or  $T^* = \mathbb{Z}\alpha$ ,  $\alpha = 2\omega$  is the simple root and  $\omega$  is the fundamental weight. The Weyl group  $W = \{1, s\}$  acts by  $s: \omega \mapsto -\omega$ , where  $s$  is the simple reflection. By definition,  $\mathbf{S} = \mathbf{R}[[x]]_F$  (where  $x = x_\omega$  or  $x = x_\alpha$ ),  $\mathbf{Q} = \mathbf{S}[\frac{1}{x}]$ ,  $\mathbf{Q}_W = \{q(x)\delta_w \mid q(x) \in \mathbf{Q}, w \in W\}$  with

$$q(-_F x)\delta_s = s(q(x))\delta_s = \delta_s q(x),$$

where  $-_F x$  is the formal inverse of  $x$ . Observe that  $x_\alpha = x_{\omega+\omega} = F(x_\omega, x_\omega)$  in  $\mathbf{S}$ .

The  $\mathbf{R}$ -algebra  $\mathbf{D}_F$  is a free left  $\mathbf{S}$ -submodule of rank 2 in  $\mathbf{Q}_W$  with basis

$$\{1, Y = \frac{1}{-_F x_\alpha} + \frac{1}{x_\alpha} \delta_s\}.$$

It satisfies the relations

$$Y^2 = \kappa Y \text{ and } Yq(x) = q(-_F x)Y + \Delta(q(x)),$$

where  $\kappa = \frac{1}{-_F x_\alpha} + \frac{1}{x_\alpha}$  and  $\Delta(q(x)) = \frac{q(x) - q(-_F x)}{-_F x_\alpha}$ .

Let  $p = a + bY$ , where  $a, b \in \mathbf{S}$ , be an idempotent in  $\mathbf{D}_F$ , i.e.,  $p^2 = p$ . Then we obtain in  $\mathbf{S}$

$$(10) \quad a^2 + b\Delta(a) = a \quad \text{and} \quad (a + s(a) + s(b)\kappa + \Delta(b))b = b.$$

In the case  $\mathbf{h}(-) = CH(-; \mathbb{Z})$  ( $F$  is additive) we have  $\mathbf{R} = \mathbb{Z}$ ,  $\mathbf{S} = \mathbb{Z}[x]$ ,  $\kappa = 0$ ,  $-_F x = -x$ ,  $x_\alpha = 2x_\omega$  and the second equation turns into

$$(a + s(a) + \Delta(b))b = b.$$

If  $x = x_\alpha$  ( $G = PGL_2$ ), the polynomials  $a + s(a)$  and  $\Delta(b)$  are divisible by 2, so  $b = 0$ . Hence,  $a = 0$  or 1 from the first equation. So  $\mathbf{D}_F$  is an indecomposable module over itself. By Theorem 8.1 this implies that both graded and non-graded motives  $[E/B]$  and  $[E/B]_*$  of a generic conic are indecomposable.

If  $x = x_\omega$  ( $G = SL_2$ ) and  $p$  is homogeneous, we get  $a \in \mathbb{Z}$ ,  $b = cx$ ,  $c \in \mathbb{Z}$  and the system (10) has solutions only for  $c = \pm 1$ . Therefore, the algebra  $\mathbf{D}_F$  has only two indecomposable graded submodules which correspond to the idempotents  $1 - x_\omega Y$  and  $x_\omega Y$ . In other words, the motives  $[E/B]$  and  $[E/B]_*$  split into a direct sum of two indecomposable motives.

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